

KAZHDAN-LUSZTIG POLYNOMIALS AND CANONICAL BASIS

IGOR FRENKEL, MIKHAIL KHOVANOV
AND ALEXANDER KIRILLOV, JR.

September 22, 1997

ABSTRACT. In this paper we show that the Kazhdan–Lusztig polynomials (and, more generally, parabolic KL polynomials) for the group S_n coincide with the coefficients of the canonical basis in n th tensor power of the fundamental representation of the quantum group $U_q \mathfrak{sl}_k$. We also use known results about canonical bases for $U_q \mathfrak{sl}_2$ to get a new proof of recurrent formulas for KL polynomials for maximal parabolic subgroups (geometrically, this case corresponds to Grassmanians), due to Lascoux–Schützenberger and Zelevinsky.

1. REVIEW OF THE THEORY OF KAZHDAN-LUSZTIG POLYNOMIALS

In this section, we review the theory of Kazhdan–Lusztig polynomials. We will use their generalization to the parabolic case, defined by Deodhar (see [D]). For the sake of completeness and to fix notations, we list the main definitions and results here, referring the reader to the original papers for more details. To avoid confusion with the theory of quantum groups, we will not use the variable q in the definition of the Hecke algebra; instead, we will use $v = q^{-1/2}$.

Let W be a finite Weyl group with a set of simple reflections S . We denote by $l(w)$ the length of $w \in W$ with respect to the generators $s \in S$.

Let $\mathcal{H}(W)$ be the Hecke algebra associated with W (we will usually omit W and write just \mathcal{H}). By definition, it is an associative algebra with unit over the field $\mathbb{Q}(v)$ with generators $T_s, s \in S$ and relations

$$(1.1) \quad \begin{aligned} T_s T_{s'} \cdots &= T_{s'} T_s \cdots && n \text{ terms on each side} \\ &\text{where } n \text{ is the order of } ss' \text{ in } W \\ (T_s + 1)(T_s - v^{-2}) &= 0. \end{aligned}$$

For any subset $J \subset S$, let W_J be the parabolic subgroup in W generated by $s \in J$. Denote by W^J the set of minimal length representatives of the cosets in W/W_J , and let $<$ be the partial order on W^J induced by the Bruhat order on W .

Let M be a vector space over $\mathbb{Q}(v)$ with the basis $m_y, y \in W^J$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

Proposition 1.1(see [D, Lemmas 2.1, 2.2]). *Let u be either -1 or v^{-2} .*

(1) *The following formulas define the structure of an \mathcal{H} -module on M :*

$$(1.2) \quad T_s m_\sigma = \begin{cases} v^{-1} m_{s\sigma} + (v^{-2} - 1) m_\sigma, & l(s\sigma) < l(\sigma), \\ v^{-1} m_{s\sigma} & l(s\sigma) > l(\sigma), \quad s\sigma \in W^J \\ u m_\sigma, & l(s\sigma) > l(\sigma), \quad s\sigma \notin W^J \end{cases}$$

Note that if $\sigma \in W^J, l(s\sigma) < l(\sigma)$ then $s\sigma \in W^J$.

(2) *If $\sigma \in W^J$ then $T_\sigma m_e = v^{-l(\sigma)} m_\sigma$.*

Remarks. 1. Our notations slightly differ from those of Deodhar: what we denote m_σ in his notations would be $q^{-l(\sigma)/2} m_\sigma$. Note also that there is a misprint in the formula for $L(s)$ (immediately after the statement of Lemma 2.1) in [D].

2. It is easy to see that M is a deformation of the induced representation $\text{Ind}_{W_J}^W 1$ of W . In particular, if $J = \emptyset$ then M is the left regular representation of \mathcal{H} and the action is independent of u .

Define an involution $\overline{} : \mathbb{Q}(v) \rightarrow \mathbb{Q}(v)$ by $f(v) \mapsto f(v^{-1})$. We will say that a map ϕ of $\mathbb{Q}(v)$ -vector spaces is antilinear if $\phi(fx) = \bar{f}\phi(x)$ for any $f \in \mathbb{Q}(v)$ and any vector x .

We define antilinear involutions on \mathcal{H} and M by letting $\overline{T_s} = T_s^{-1} = v^2 T_s + v^2 - 1$, $\overline{m_e} = m_e$ and $\overline{ab} = \bar{a}\bar{b}$. Note that one has $\overline{m_\sigma} = (v^{l(\sigma)} T_\sigma) m_e = m_\sigma + \sum_{\tau < \sigma} c_{\tau\sigma} m_\tau$.

Theorem 1.2([D, Proposition 3.2]). *Let us assume that we have fixed u as in Proposition 1.1. Then for every $\sigma \in W^J$ there exists a unique element $C_\sigma \in M$ such that the following two conditions are satisfied:*

$$(1.3) \quad \begin{aligned} \overline{C_\sigma} &= C_\sigma, \\ C_\sigma &= \sum_{\substack{\tau \in W^J \\ \tau \leq \sigma}} \alpha_{\tau\sigma} m_\tau, \\ &\text{where } \alpha_{\sigma\sigma} = 1 \quad \text{and } \alpha_{\tau\sigma} \in v^{-1}\mathbb{Z}[v^{-1}] \quad \text{for } \tau < \sigma. \end{aligned}$$

The elements $C_\sigma, \sigma \in W^J$ form a basis in M .

Note that the definition of C_σ uses the involution $\overline{}$, which was defined in terms of the action of \mathcal{H} and therefore depends on the choice of u . Thus, we have two different antilinear involutions on the same space M with fixed basis m_y , which give rise to different bases C_σ .

Following Deodhar, we define **parabolic Kazhdan-Lusztig polynomials** $P_{\tau\sigma}^J \in \mathbb{Z}[q], \tau, \sigma \in W^J, \tau \leq \sigma$ by

$$(1.4) \quad \alpha_{\tau\sigma} = (-v)^{l(\tau)-l(\sigma)} \overline{P_{\tau\sigma}^J}$$

where $\alpha_{\tau\sigma}$ is defined by (1.3) and as before, $q = v^{-2}$. This polynomials can be expressed via the usual KL polynomials as follows:

Theorem 1.3 [D, Proposition 3.4 and Remark 3.8].

(1) For $u = -1$ we have

$$(1.5) \quad P_{\tau\sigma}^J = P_{\tau w_J^0, \sigma w_J^0},$$

where w_J^0 is the longest element in W_J and $P_{y,w}$ is the usual Kazhdan-Lusztig polynomial for W .

(2) For $u = v^{-2}$,

$$(1.6) \quad P_{\tau\sigma}^J = \sum_{\substack{w \in W_J \\ \tau w \leq \sigma}} (-1)^{l(w)} P_{\tau w, \sigma}.$$

In this paper we will be interested in the case $W = S_n$. We will consider S_n as the group acting by permutations on the set of all integer sequences of length n , with the generators $s_i, i = 1, \dots, n-1$ acting in the standard way: $s_i(x_1, \dots, x_n) = (x_1, \dots, x_{i+1}, x_i, \dots, x_n)$.

2. CANONICAL BASIS FOR $U_q \mathfrak{sl}_k$.

In this section, we show that the parabolic Kazhdan-Lusztig basis for S_n can be obtained as a special case of the canonical basis for representations of quantum groups. This is inspired by the results of [GL], where it is proved that the projectivization of the KL basis coincides with the so-called “special basis” for quantum groups. However, it would take us more time to define what a special basis is and why it is the projectivization of the canonical basis. For this reason, we are using the results of [GL] only as motivation.

Let us fix an integer $k \geq 2$. Let $U_q \mathfrak{sl}_k$ be the quantum group corresponding to the Lie algebra \mathfrak{sl}_k , that is, an associative algebra over the field $\mathbb{Q}(v)$ with generators $E_i, F_i, v^{H_i}, i = 1, \dots, k-1$ and standard relations (see, for example, [L], where v^{H_i} is denoted by K_i). We denote by $(U_q \mathfrak{sl}_k)^\pm$ subalgebras in $U_q \mathfrak{sl}_k$ generated by E_i, v^{H_i} (respectively, by F_i, v^{H_i}). We identify the weight lattice P for \mathfrak{sl}_k with $\mathbb{Z}^k / \mathbb{Z} \cdot (1, \dots, 1)$ by the rule $H_i(\lambda) = \lambda_i - \lambda_{i+1}, \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k$.

Let V be a k -dimensional vector space over $\mathbb{Q}(v)$ with the basis $\mathbf{e}_1, \dots, \mathbf{e}_k$. We define on V the structure of a representation of $U_q \mathfrak{sl}_k$ by

$$(2.1) \quad \begin{aligned} E_i \mathbf{e}_{i+1} &= \mathbf{e}_i, & E_i \mathbf{e}_j &= 0, & j &\neq i+1, \\ F_i \mathbf{e}_i &= \mathbf{e}_{i+1}, & F_i \mathbf{e}_j &= 0, & j &\neq i, \\ v^{H_i} \mathbf{e}_j &= \begin{cases} v \mathbf{e}_i, & j = i, \\ v^{-1} \mathbf{e}_{i+1}, & j = i+1, \\ \mathbf{e}_i, & j \neq i, i+1 \end{cases} \end{aligned}$$

As is well-known (see, e.g., [L], [CP]), $U_q \mathfrak{sl}_k$ can be endowed with the structure of a Hopf algebra with the comultiplication

$$(2.2) \quad \begin{aligned} \Delta E_i &= E_i \otimes 1 + v^{H_i} \otimes E_i, \\ \Delta F_i &= F_i \otimes v^{-H_i} + 1 \otimes F_i, \\ \Delta v^{H_i} &= v^{H_i} \otimes v^{H_i} \end{aligned}$$

We do not write formulas for the antipode and counit since we will not use them.

Moreover, $U_q \mathfrak{sl}_k$ is quasitriangular: there is an element \mathcal{R} (universal R -matrix) in a certain completion of $(U_q \mathfrak{sl}_k)^{\otimes 2}$, which, among other properties, has the following one. For any pair of finite-dimensional representations V, W with weight decomposition, $R_{V \otimes W} = \mathcal{R}|_{V \otimes W}$ is well-defined and $PR_{V \otimes W} : V \otimes W \rightarrow W \otimes V$ is an isomorphism of $U_q \mathfrak{sl}_k$ -modules. In fact, this last property uniquely defines \mathcal{R} if we also require that it have the following form:

$$(2.3) \quad \mathcal{R} = \bar{C} \bar{\Theta}$$

where

$$(2.4) \quad \begin{aligned} \bar{C} &= v^{\sum x_a \otimes x_a}, \\ \bar{\Theta} &= 1 + \sum a_i \otimes b_i, \quad a_i \in (U_q \mathfrak{sl}_k)^-, b_i \in (U_q \mathfrak{sl}_k)^+ \end{aligned}$$

where x_a is an orthonormal basis in the Cartan subalgebra of \mathfrak{sl}_k and a_i (respectively, b_i) are of negative (respectively, positive) weight. We refer the reader to [L] or [CP] for details.

Remark. The notations $\bar{C}, \bar{\Theta}$ are chosen to agree with notations of Lusztig and in [FK]. Later we will define the bar involution $\bar{}$ and elements C, Θ so that $\bar{C}, \bar{\Theta}$ defined by (2.4) will indeed be the bar conjugates of C, Θ .

Let us consider the vector space $V^{\otimes n}$. This space has a basis given by

$$(2.5) \quad \mathbf{e}_I = \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_n}, \quad I = (i_1, \dots, i_n) \in \{1, \dots, k\}^n.$$

We define the action of $U_q \mathfrak{sl}_k$ on $V^{\otimes n}$ using the comultiplication Δ . This space has a weight decomposition: if $\mathbf{m} \in \mathbb{Z}_+^k$ is such that $\sum m_i = n$ then a basis in the weight subspace $V^{\otimes n}[\mathbf{m}]$ is given by the vectors \mathbf{e}_I such that every $a = 1, \dots, k$ appears in the sequence $I = (i_1, \dots, i_n)$ exactly m_a times.

Proposition 2.1.

- (1) Let $P : V \otimes V \rightarrow V \otimes V$ be the permutation: $P(v \otimes w) = w \otimes v$, and let $R_{V \otimes V}$ denote the action of the universal R -matrix \mathcal{R} in $V \otimes V$. Then

$$(2.6) \quad PR_{V \otimes V}(\mathbf{e}_a \otimes \mathbf{e}_b) = v^{-1/n} \begin{cases} \mathbf{e}_b \otimes \mathbf{e}_a + (v - v^{-1})\mathbf{e}_a \otimes \mathbf{e}_b, & a < b, \\ \mathbf{e}_b \otimes \mathbf{e}_a, & a > b, \\ v\mathbf{e}_a \otimes \mathbf{e}_b, & a = b \end{cases}$$

- (2) Denote by $(PR)_i$ an endomorphism of $V^{\otimes n}$ which acts as $PR_{V \otimes V}$ on the tensor product of i -th and $i+1$ -th factors and by identity on all other factors. Then the map

$$(2.7) \quad T_i \mapsto -v^{-1+1/n}(PR)_i$$

defines an action of the Hecke algebra $\mathcal{H}(S_n)$ on $V^{\otimes n}$. This action commutes with the action of $U_q \mathfrak{sl}_k$; in particular, it preserves the weight subspaces.

(3) For $\mathbf{m} \in \mathbb{Z}_+^k$, $\sum m_i = n$ let

$$(2.8) \quad I^0(\mathbf{m}) = (\underbrace{k, \dots, k}_{m_k \text{ times}}, \dots, \underbrace{1, \dots, 1}_{m_1 \text{ times}}).$$

Let $J \subset \{1, \dots, n-1\}$ be such that the corresponding parabolic subgroup $W_J \subset S_n$ is the stabilizer of $I^0(\mathbf{m})$, i.e. $W_J = S_{m_k} \times \dots \times S_{m_1}$, and let M be the corresponding vector space as defined in Section 1. Then the map

$$(2.9) \quad \begin{aligned} \phi_{\mathbf{m}} : M &\rightarrow V^{\otimes n} \\ m_{\sigma} &\mapsto (-1)^{l(\sigma)} \mathbf{e}_{\sigma(I^0(\mathbf{m}))} \end{aligned}$$

is an isomorphism of \mathcal{H} modules M (with $u = -1$) and $V^{\otimes n}[\mathbf{m}]$.

Proof. Parts (1), (2) are due to Jimbo [J] and by now are well-known. As for (3), it follows from the comparison of formulas (1.2) and (2.6) and the following simple lemma, the proof of which is left to the reader.

Lemma. Let \mathbf{m}, J be as in the statement of the theorem, and let $\sigma \in W^J$. Let a and b be i -th and $i+1$ -th entries of $\sigma(I^0(\mathbf{m}))$, respectively. Then $l(s_i \sigma) > l(\sigma)$ if and only if $a \geq b$, and $s_i \sigma \in W^J$ if and only if $a \neq b$.

□

For future reference, we formulate the following result, which is very close to the lemma above.

Proposition 2.2. Let \mathbf{m}, J be as in the previous proposition. Then the map

$$\sigma \mapsto \sigma(I^0(\mathbf{m}))$$

is a bijection between W^J and the set of all sequences of weight \mathbf{m} , and

$$l(\sigma) = \{I^0(\mathbf{m})\} - \{\sigma(I^0(\mathbf{m}))\}$$

where $\{I\}$ is the number of inversions in I , i.e. the number of pairs of indices a, b such that $a < b$ and $i_a > i_b$.

Recall that we define the bar involution on $\mathbb{Q}(v)$ by $\bar{v} = v^{-1}$ and we say that a map ϕ of $\mathbb{Q}(v)$ vector spaces is antilinear if $\phi(fv) = \bar{f}\phi(v)$. Following Lusztig, we will also denote by a bar the antilinear algebra automorphism of $U_q \mathfrak{sl}_k$ defined on the generators by

$$\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{v^{H_i}} = v^{-H_i}.$$

We extend the bar involution to $U_q \mathfrak{sl}_k \otimes U_q \mathfrak{sl}_k$ by the rule $\overline{\bar{x} \otimes \bar{y}} = \bar{x} \otimes \bar{y}$ and denote by C, Θ the bar conjugates of the elements $\bar{C}, \bar{\Theta}$ defined by (2.4). We will use the following result due to Lusztig: $\Theta \bar{\Theta} = 1$. Note also that $C \bar{C} = 1$ (obvious).

Let us define by induction an antilinear involution ψ on tensor powers of the module V as follows:

(1) On V , ψ is given by

$$(2.10a) \quad \psi \left(\sum a_i \mathbf{e}_i \right) = \sum \overline{a_i} \mathbf{e}_i.$$

(2) If W_1, W_2 are tensor powers of V and the involution ψ is already defined on W_1, W_2 then on $W_1 \otimes W_2$ we have

$$(2.10b) \quad \psi(w_1 \otimes w_2) = \Theta(\psi(w_1) \otimes \psi(w_2)).$$

As before, the action of $U_q \mathfrak{sl}_k$ (and thus, of Θ) on W_i is defined using the comultiplication Δ .

It follows from the results of Lusztig that ψ is well defined on any tensor power of V and that $\Psi^2 = 1$.

We will use the involution ψ to define the canonical basis in $V^{\otimes n}$. The following proposition is a special case of the definition in [L, Chapter 27].

Proposition 2.3. *For every $I \in \{1, \dots, k\}^n$ there exists a unique element $b_I \in V^{\otimes n}$ such that*

$$(2.11) \quad \begin{aligned} \psi(b_I) &= b_I, \\ b_I - \mathbf{e}_I &\in \bigoplus_{I'} v^{-1} \mathbb{Z}[v^{-1}] \mathbf{e}_{I'}, \end{aligned}$$

where the sum is taken over all I' having the same weight as I .

The elements b_I form a basis in $V^{\otimes n}$ which is called the **canonical basis**.

The first important result of this section is the following proposition.

Proposition 2.4. *The isomorphism $\phi_{\mathbf{m}}$ constructed in Proposition 2.1 identifies the bar involution in M (for $u = -1$) with the involution ψ in $V^{\otimes n}[\mathbf{m}]$.*

Proof. Since M is spanned by $T_y m_e$, it suffices to prove that $\psi(\phi_{\mathbf{m}}(m_e)) = \phi_{\mathbf{m}}(m_e)$, $\psi(\phi_{\mathbf{m}}(T_i)) = \phi_{\mathbf{m}}(T_i^{-1})\psi$, or, equivalently, $\psi(\mathbf{e}_{I^0(\mathbf{m})}) = \mathbf{e}_{I^0(\mathbf{m})}$, $\psi(PR)_i = (PR)_i^{-1}\psi$.

To prove the first identity, let $<$ be the partial order on $\{1, \dots, k\}^n$ obtained by the transitive closure of

$$(\dots a \dots b \dots) < (\dots b \dots a \dots) \quad \text{if } a > b$$

Then it is easy to show, using (2.4), that $\psi(\mathbf{e}_I) = \mathbf{e}_I + \sum_{I' < I} c_{I,I'} \mathbf{e}_{I'}$. On the other hand, $I^0(\mathbf{m})$ is a minimal element with respect to this order, which proves $\psi(\mathbf{e}_{I^0(\mathbf{m})}) = \mathbf{e}_{I^0(\mathbf{m})}$.

To prove that $\psi(PR)_i = (PR)_i^{-1}\psi$, let us first consider the case $n = 2$. Then this identity reduces to $\Theta \overline{P \bar{C} \bar{\Theta}} = (P \bar{C} \bar{\Theta})^{-1} \Theta$, which is obvious in view of $\bar{C} = C^{-1}$, $\bar{\Theta} = \Theta^{-1}$.

For $n \geq 3$, the identity $\psi(PR)_i = (PR)_i^{-1}\psi$ follows by induction from the $n = 2$ case and the fact that $(PR)_i$ is an intertwiner. Indeed, assume that W is a tensor

power of V and $T : W \rightarrow W$ is an intertwining operator such that $\psi T = T^{-1}\psi$. Then the same is true for the operators $1 \otimes T : V \otimes W \rightarrow V \otimes W, T \otimes 1 : W \otimes V \rightarrow W \otimes V$. This is because the involution ψ on $V \otimes W$ is given by $\psi = \Theta(\psi \otimes \psi)$, and therefore $\psi(1 \otimes T) = \Theta(\psi \otimes (\psi T)) = \Theta(\psi \otimes (T^{-1}\psi)) = (1 \otimes T^{-1})\Theta(\psi \otimes \psi) = (1 \otimes T^{-1})\psi$. \square

Theorem 2.5. *Under the assumptions of Proposition 2.1, the isomorphism $\phi_{\mathbf{m}}$ maps the Kazhdan-Lusztig basis in M defined in Theorem 1.2 (for $u = -1$) to the canonical basis in $V^{\otimes n}[\mathbf{m}]$:*

$$\phi_{\mathbf{m}}(C_{\sigma}) = (-1)^{l(\sigma)} b_{\sigma(I^0(\mathbf{m}))}$$

Proof. Immediately follows from the definitions and from the previous proposition. \square

So far, we have discussed the relation between the basis C_{σ} in M defined for $u = -1$ and the canonical basis for $U_q \mathfrak{sl}_k$. It turns out that the basis C_{σ} for $u = v^{-2}$ also admits a nice interpretation in terms of representations of $U_q \mathfrak{sl}_k$: it is related with the so-called dual canonical basis.

Let us define on $U_q \mathfrak{sl}_k$ another structure of Hopf algebra by

$$(2.12) \quad \begin{aligned} \bar{\Delta} E_i &= E_i \otimes 1 + v^{-H_i} \otimes E_i, \\ \bar{\Delta} F_i &= F_i \otimes v^{H_i} + 1 \otimes F_i, \\ \bar{\Delta} v^{H_i} &= v^{H_i} \otimes v^{H_i} \end{aligned}$$

The universal R -matrix for this comultiplication is given by

$$(2.13) \quad \bar{R} = C\Theta.$$

From now on, let us consider the action of $U_q \mathfrak{sl}_k$ on tensor powers of V given by iterations of $\bar{\Delta}$. Note that since $\bar{\Delta}(v^{H_i}) = \Delta(v^{H_i})$, the weight decomposition for both actions coincide. Define an antilinear involution ψ' on tensor powers of V by the same formulas as (2.10) but replacing Θ by $\bar{\Theta}$, which acts on tensor powers of V via $\bar{\Delta}$.

Then the propositions 2.1–2.4 have their analogue. We formulate the corresponding statements below, using the same notations and conventions as much as possible.

Proposition 2.1'. *In the notations of Proposition 2.1, we have*

(1)

$$(2.14) \quad P\bar{R}_{V \otimes V}(\mathbf{e}_a \otimes \mathbf{e}_b) = v^{1/n} \begin{cases} \mathbf{e}_b \otimes \mathbf{e}_a + (v^{-1} - v)\mathbf{e}_a \otimes \mathbf{e}_b, & a < b, \\ \mathbf{e}_b \otimes \mathbf{e}_a, & a > b, \\ v^{-1}\mathbf{e}_a \otimes \mathbf{e}_b, & a = b \end{cases}$$

(2)

The map

$$(2.15) \quad T_i \mapsto v^{-1-1/n}(P\bar{R})_i$$

defines an action of the Hecke algebra $\mathcal{H}(S_n)$ on $V^{\otimes n}$. This action commutes with the action of $U_q\mathfrak{sl}_k$; in particular, it preserves the weight subspaces.

(3) For $\mathbf{m} \in \mathbb{Z}_+^k$, $\sum m_i = n$ the map

$$(2.16) \quad \begin{aligned} \phi'_{\mathbf{m}} : M &\rightarrow V^{\otimes n} \\ m_\sigma &\mapsto \mathbf{e}_{\sigma(I^0(\mathbf{m}))} \end{aligned}$$

is an isomorphism of \mathcal{H} modules M (with $u = v^{-2}$) and $V^{\otimes n}[\mathbf{m}]$.

Proposition 2.3'([FK]).

(1) For every $I \in \{1, \dots, k\}^n$ there exists a unique element $b^I \in V^{\otimes n}$ such that

$$(2.17) \quad \begin{aligned} \psi'(b^I) &= b^I, \\ b^I - \mathbf{e}_I &\in \bigoplus_{I'} v^{-1}\mathbb{Z}[v^{-1}]\mathbf{e}_{I'}, \end{aligned}$$

where the sum is taken over all I' having the same weight as I .

(2) Denote by \langle, \rangle the $\mathbb{Q}(v)$ -bilinear form on $V^{\otimes n}$ defined by

$$(2.18) \quad \langle \mathbf{e}_I, \mathbf{e}_{I'} \rangle = \delta_{I, w_0(I')},$$

where w_0 is the longest element in S_n : $w_0(1, \dots, n) = (n, \dots, 1)$. Then:

$$(2.19) \quad \langle b^I, b_{I'} \rangle = \delta_{I, w_0(I')}.$$

The elements b^I form a basis in $V^{\otimes n}$ which is called the **dual canonical basis**.

This is the only proposition which requires a separate proof. It does not immediately follow from Proposition 2.3 since we have replaced v by v^{-1} in all the formulas but left condition (2.11) unchanged. To prove the proposition, define the elements b^I by (2.19); then the second condition in (2.17) is satisfied automatically, and one only needs to check $\psi'(b^I) = b^I$, which is sufficient to check for $n = 2$. We refer the reader to [FK] for the details in the case of \mathfrak{sl}_2 ; the general case is proved in the same way.

Proposition 2.4'. The isomorphism $\phi'_{\mathbf{m}}$ constructed in Proposition 2.1' identifies the bar involution in M (for $u = v^{-2}$) with the involution ψ' in $V^{\otimes n}[\mathbf{m}]$.

Theorem 2.5'. *Under the assumptions of Proposition 2.1, the isomorphism $\phi_{\mathbf{m}}$ maps the Kazhdan-Lusztig basis in M defined in Theorem 1.2 (for $u = v^{-2}$) to the dual canonical basis in $V^{\otimes n}[\mathbf{m}]$:*

$$\phi'_{\mathbf{m}}(C_{\sigma}) = b_{\sigma(I^0(\mathbf{m}))}$$

Therefore, we see that the picture is quite symmetric: we have two actions of Hecke algebra on the module M , which give rise to two different bases C_{σ} . Similarly, we have two different actions of $U_q\mathfrak{sl}_k$ on $V^{\otimes n}$, which give rise to the canonical basis and the dual canonical basis.

In particular, let us consider the case $k = n$, and $\mathbf{m} = (1, \dots, 1)$. In this case $J = \emptyset$, and we get the following theorem:

Theorem 2.6. *The canonical basis b_I and the dual canonical basis b^I in the zero-weight subspace of $V^{\otimes n}$, where V is the fundamental representation of $U_q(\mathfrak{sl}_n)$, are given by*

$$(2.20) \quad b_{w(n, \dots, 1)} = \sum_{y \leq w} v^{l(y) - l(w)} \overline{P_{y, w}} \mathbf{e}_{y(n, \dots, 1)}$$

$$(2.21) \quad b^w(n, \dots, 1) = \sum_{y \leq w} (-v)^{l(y) - l(w)} \overline{P_{y, w}} \mathbf{e}_{y(n, \dots, 1)},$$

where $P_{y, w} = P_{y, w}(q)$, $q = v^{-2}$ are the Kazhdan-Lusztig polynomials for S_n .

Therefore, the problem of calculating Kazhdan-Lusztig polynomials can be considered as a special case of more general problem of calculating the canonical basis in a tensor power of the fundamental representation of $U_q\mathfrak{sl}_k$. This latter problem, in general, is no easier than the original one, and there is little hope that it can be solved in full generality by elementary methods. However, there is one special case in which the answer is known: this is the case of $U_q\mathfrak{sl}_2$, which we will consider in the next section.

3. CANONICAL BASIS FOR $U_q\mathfrak{sl}_2$ AND KAZHDAN-LUSZTIG POLYNOMIALS FOR GRASSMANIANS.

In this section we recall the known results about the canonical basis for tensor product of representations of $U_q\mathfrak{sl}_2$ and use the results of the previous section to transform these results into some statements about Kazhdan-Lusztig polynomials. It turns out that in this way we exactly recover the known formulas for Kazhdan-Lusztig polynomials for Grassmannians, i.e. those corresponding to maximal parabolic subgroups in S_n . These formulas were first obtained by Lascoux and Schützenberger [LS] by purely combinatorial methods and later by Zelevinsky [Z] using a small resolution of singularities of the corresponding Schubert varieties.

In this section we consider the quantum group $U_q\mathfrak{sl}_2$. It is generated by the elements $E = E_1, F = F_1, v^H = v^{H_1}$ with the usual commutation relations and with the comultiplication given by (2.2). The fundamental representation V is two dimensional: $V = \mathbb{Q}(v)\mathbf{e}_+ \oplus \mathbb{Q}(v)\mathbf{e}_-$, and the action of $U_q\mathfrak{sl}_2$ is given by

$$(3.1) \quad \begin{aligned} E\mathbf{e}_+ &= 0, & E\mathbf{e}_- &= \mathbf{e}_+, \\ F\mathbf{e}_+ &= \mathbf{e}_-, & F\mathbf{e}_- &= 0, \\ v^H\mathbf{e}_{\pm} &= v^{\pm 1}\mathbf{e}_{\pm}. \end{aligned}$$

The action of the universal R -matrix in $V \otimes V$ is given by

$$\begin{aligned}
(3.2) \quad PR(\mathbf{e}_+ \otimes \mathbf{e}_+) &= v^{1/2} \mathbf{e}_+ \otimes \mathbf{e}_+, \\
PR(\mathbf{e}_- \otimes \mathbf{e}_-) &= v^{1/2} \mathbf{e}_- \otimes \mathbf{e}_-, \\
PR(\mathbf{e}_- \otimes \mathbf{e}_+) &= v^{-1/2} \mathbf{e}_+ \otimes \mathbf{e}_-, \\
PR(\mathbf{e}_+ \otimes \mathbf{e}_-) &= v^{-1/2} \mathbf{e}_- \otimes \mathbf{e}_+ + (v^{1/2} - v^{-3/2}) \mathbf{e}_+ \otimes \mathbf{e}_-.
\end{aligned}$$

The relation with the notations of the previous section is given by $\mathbf{e}_+ = \mathbf{e}_1, \mathbf{e}_- = \mathbf{e}_2$.

In this case it is possible to give an explicit construction of the dual canonical basis in the tensor power $V^{\otimes n}$ (and, in fact, in any tensor product of finite-dimensional representations of $U_q \mathfrak{sl}_2$). This was done in [FK]. We give here the answer in the case of interest for us.

Let

$$(3.3) \quad a = \mathbf{e}_+ \otimes \mathbf{e}_- - v^{-1} \mathbf{e}_- \otimes \mathbf{e}_+ \in V \otimes V.$$

One easily check that a is an invariant: $\bar{\Delta}E(a) = \bar{\Delta}(F)(a) = 0$ and that $\psi'(a) = a$. Thus, a is an element of the dual canonical basis in $V^{\otimes 2}[0]$.

The following theorem describes the dual canonical basis in tensor powers $V^{\otimes m}$. This basis is naturally indexed by sequences I of pluses and minuses of length m (as before, to relate to the notation of the previous section, replace plus by 1 and minus by 2). If I_1, I_2 are sequences of length l, k respectively then we will denote by $I_1|I_2$ the sequence of length $k+l$ obtained by appending the sequence I_2 to the sequence I_1 .

Theorem 3.1. *The dual canonical basis b^I in $V^{\otimes m}$ is given by the following rules:*

- (1) *If $I = -|I_1$ then $b^I = \mathbf{e}_- \otimes b^{I_1}$.*
- (2) *If $I = I_1|+$ then $b^I = b^{I_1} \otimes \mathbf{e}_+$.*
- (3) *If $I = I_1|+|-|I_2$, where I_1, I_2 are sequences of length $i, m-i-2$ respectively then*

$$(3.4) \quad b^I = a_{i+1i+2} b^{I_1|I_2},$$

where the operator $a_{i+1i+2} : V^{\otimes(m-2)} \rightarrow V^{\otimes m}$ is defined by

$$(3.5) \quad a_{i+1i+2} : \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_{m-2}} \mapsto \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_i} \otimes a \otimes \mathbf{e}_{j_{i+1}} \otimes \dots \otimes \mathbf{e}_{j_{m-2}}.$$

Here $a \in V \otimes V$ is given by (3.3).

As was mentioned above, this theorem is a special case of the results in [FK, Section 2.3]; however, this case is not difficult to prove from the definitions. We leave it as an exercise for the reader. This theorem can be very neatly visualized using the graphical calculus; this interpretation can be found in [FK].

Example 3.2.

$$\begin{aligned} b^{++--} &= \mathbf{e}_+ \otimes a \otimes \mathbf{e}_- - v^{-1} \mathbf{e}_- \otimes a \otimes \mathbf{e}_+ \\ &= \mathbf{e}_{++--} - v^{-1} \mathbf{e}_{-++-} - v^{-1} \mathbf{e}_{+-+-} + v^{-2} \mathbf{e}_{--++}. \end{aligned}$$

Since we have proved that the dual canonical basis in $V^{\otimes n}$ coincides with the parabolic Kazhdan-Lusztig basis C_σ defined in Theorem 1.2 (for $u = v^{-2}$), the above theorem also gives a very explicit way to calculate the latter basis for the maximal parabolic subgroup in S_n . In particular, we have the following corollary.

Corollary 3.3. *In the notations of Section 1, assume that J is a maximal parabolic subgroup in $W = S_n$. Then each parabolic Kazhdan-Lusztig polynomial $P_{\tau\sigma}^J$ (for $u = v^{-2}$) is either zero or a power of q .*

Proof. Immediately follows from Theorems 3.1, 2.5' and the explicit formula (3.3) for a . \square

We next describe the canonical basis in tensor powers $V^{\otimes m}$. In the same way as for the dual canonical basis, we will index the basis vectors by the length m sequences of pluses and minuses. For a sequence J of pluses and minuses denote by $l(J)$ the length of J and by J_+ the number of pluses in J .

Let $p_n : V^{\otimes n} \rightarrow V^{\otimes n}$ be the operator of the projection onto the unique irreducible $(n+1)$ -dimensional submodule of $V^{\otimes n}$ (relative to the comultiplication Δ). This operator is sometimes called **the Jones-Wenzl projector**. Coordinatewise p_n is just the q -symmetrisation: let $I = i_1 \dots i_n$ be a sequence which contains k pluses and $n - k$ minuses (in any order). Then

$$(3.6) \quad p_n(\mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_n}) = \left[\begin{matrix} n \\ k \end{matrix} \right]^{-1} \sum_{J=j_1 \dots j_n, J_+=k} v^{k(n-k)-\{I\}-\{J\}} \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_n}$$

The sum in the right hand side is over all sequences of length n that have the same number of pluses as the sequence $i_1 \dots i_n$, and $\{J\}$ is the number of pairs $(a, b), 1 \leq a < b \leq n$ such that $j_a = -$ and $j_b = +$ (compare with Proposition 2.2).

The quantum binomial coefficient is defined by

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \frac{[n]!}{[k]![n-k]!},$$

where $[n]! = [1][2] \dots [n]$ and $[i] = \frac{v^i - v^{-i}}{v - v^{-1}}$.

Also let $p_{i,j} : V^{\otimes m} \rightarrow V^{\otimes m}$ be given by

$$(3.7) \quad p_{i,j} = 1^{\otimes(i-1)} \otimes p_{j-i+1} \otimes 1^{\otimes(m-j)}$$

Theorem 3.4. *The canonical basis b_I in $V^{\otimes m}$ is given by the following rules:*

- (1) *If $I = -|I_1$ then $b_I = \mathbf{e}_- \otimes b_{I_1}$.*
- (2) *If $I = I_1|+$ then $b_I = b_{I_1} \otimes \mathbf{e}_+$.*
- (3) *Let $I = I_1|I_+|I_-|I_2$, where I_+ is made entirely of pluses, I_- entirely of minuses, and the following conditions are satisfied:*

- $l(I_+) = k > 0$, $l(I_-) = l > 0$
 - I_1 is either empty or ends with at least k minuses.
 - I_2 is either empty or starts with at least l pluses.
- Then

$$(3.8) \quad b_I = \begin{bmatrix} k+l \\ l \end{bmatrix} p_{l(I_1)+1, l(I_1)+k+l} b_{I_1|I_-|I_+|I_2},$$

This theorem is proved in [K]. The proof is based on Theorem 3.1, describing the dual canonical basis, and the identity $\langle b_I, b^{I'} \rangle = \delta_{I, w_0(I')}$ (see (2.19)).

This theorem describes each canonical basis vector as a certain composition of Jones-Wenzl projectors applied to the vector $\mathbf{e}_- \otimes \dots \otimes \mathbf{e}_- \otimes \mathbf{e}_+ \otimes \dots \otimes \mathbf{e}_+$.

Example. $b_{+-++-} = [2][4]p_{1,2}p_{2,5}\mathbf{e}_{--++-}$.

Note that for some canonical basis vectors such a representation is not unique:

$$b_{+-++-} = [2][3]p_{1,2}p_{2,4}\mathbf{e}_{--++-} = [2][3]p_{3,4}p_{1,3}\mathbf{e}_{--++-}.$$

By Theorem 2.4, this theorem also describes the Kazhdan-Lusztig basis in M for $u = -1$. Thus, in this case the Kazhdan-Lusztig basis vectors can be expressed as compositions of Jones-Wenzl projectors.

Next we show that if the recursive formulas of Theorem 3.4 are written coordinatewise, one obtains the recursive formulas of A. Lascoux, M.-P. Schützenberger [LS] and Zelevinski [Z] for Kazhdan-Lusztig polynomials in the Grassmannian case.

For a pair of sequences $I, J \in \{+, -\}^n$ denote by $c(I, J)$ the coefficient of \mathbf{e}_J in the decomposition of b_I in the product basis of $V^{\otimes n}$. Order the set $\{+, -\}$ by $+ > -$. Let $I = i_1 \dots i_n$ and $J = j_1 \dots j_n$. Theorem 3.4 implies that if $i_a \geq i_{a+1}$ then $p_{a,a+1}b_I = b_I$. Therefore, if $i_a \geq i_{a+1}$, and $j_a = -, j_{a+1} = +$, then

$$(3.9) \quad c(I, J) = v^{-1}c(I, J^{(a)}),$$

where $J^{(a)}$ is the sequence obtained by interchanging the a th and $(a+1)$ st elements of J . Thus we can reduce the computation of $c(I, J)$ to the case $j_a \geq j_{a+1}$ whenever $i_a \geq i_{a+1}$. If J has this property, we say that J is controlled by I .

Before we proceed, we introduce new notations to match the ones in [Z]. We can encode I by a sequence of numbers $b_0, a_1, b_1, a_2, \dots, b_{m-1}, a_m$ as follows. I starts with b_0 pluses, followed by a_1 minuses, followed by b_1 pluses, etc. I fixed, we encode J (where J is controlled by I) relative to I by a sequence x_0, \dots, x_{m-1} . Namely, J begins with x_0 pluses, followed by $b_0 + a_1 - x_0$ minuses, followed by x_1 pluses, followed by $b_1 + a_2 - x_1$ minuses, etc.

We want a recursive formula for $c(I, J)$ in terms of these encodings of I and J . We apply Theorem 3.4. Suppose we can find i such that $a_i \geq b_i$ and $a_{i+1} \leq b_{i+1}$ (this is the most interesting case; in the notations of Theorem 3.4 it corresponds to case when both I_1, I_2 are non-empty). Denote by L the sequence obtained by interchanging two subsequences of I : the subsequence of b_i pluses and the adjacent subsequence of a_{i+1} minuses. L is encoded by $b_0, a_1, \dots, b_{i-1}, a_i + a_{i+1}, b_i + b_{i+1}, a_{i+2}, b_{i+2}, \dots, a_m$.

By Theorem 3.4
(3.10)

$$b_I = \begin{bmatrix} a_{i+1} + b_i \\ b_i \end{bmatrix} p_{k+1, k+a_{i+1}+b_i} b_L = \begin{bmatrix} a_{i+1} + b_i \\ b_i \end{bmatrix} p_{k+1, k+a_{i+1}+b_i} \sum_{J'} c(L, J') \mathbf{e}_{J'},$$

where $k = b_0 + a_1 + \dots + b_{i-1} + a_i$.

Since the projector $p_{k+1, k+a_{i+1}+b_i}$ can only change the indices of $\mathbf{e}_{J'}$ which are in the interval $[k+1, k+1+a_{i+1}+b_i]$, the only terms in (3.10) which give non-zero contribution to $c(I, J)$ are those with J' coinciding with J outside of this interval. Using (3.9), we replace each such J' by a sequence controlled by L , which together with the explicit formula (3.6) for the projector gives

$$(3.11) \quad c(I, J) = \sum_s \begin{bmatrix} x_i \\ s \end{bmatrix} \begin{bmatrix} a_{i+1} + b_i - x_i \\ a_{i+1} - s \end{bmatrix} v^{f(s)} c(L, J_s)$$

where J_s is the sequence, controlled by L , and encoded, relative to L , by

$$x_0, \dots, x_{i-2}, x_{i-1} + s, x_i + x_{i+1} - s, x_{i+2}, \dots, x_{m-1},$$

and

$$(3.12) \quad f(s) = s(b_i - x_i) + s^2 - s(a_i + b_{i-1} - x_{i-1}) - x_{i+1}(b_i - x_i + s).$$

Introducing new parameters

$$\begin{aligned} c_0 &= x_0, c_{j+1} = c_j + b_j - x_j, \quad 1 \leq j \leq m, \\ d &= c_i - s \end{aligned}$$

we get

$$(3.13) \quad c(I, J) = \sum_s \begin{bmatrix} b_i - c_{i+1} + c_i \\ c_i - d \end{bmatrix} \begin{bmatrix} a_{i+1} + c_{i+1} - c_i \\ c_{i+1} - d \end{bmatrix} v^{f(s)} c(L, J_s)$$

and $f(s)$ can be rewritten as

$$f(s) = (c_i - d)b_i + (c_{i+1} - d)a_{i+1} + g(s)$$

where

$$\begin{aligned} g(s) &= -((c_i - d)(a_i + b_i) + (c_{i+1} - d)(b_{i+1} + a_{i+1})) + \\ &= c_{i-1}c_i - c_i^2 + c_i c_{i+1} - c_{i+1}^2 + c_{i+1}c_{i+2} - dc_{i-1} - dc_{i+2} + d^2 \end{aligned}$$

Now let

$$(3.14) \quad h(I, J) = - \sum_j c_j c_{j+1} + \sum_j c_j (c_j + a_j + b_j)$$

and

$$(3.15) \quad c^0(I, J) = v^{h(I, J)} c(I, J).$$

Then the formula (3.13) becomes

$$(3.16) \quad c^0(I, J) = \sum_s \begin{bmatrix} b_i - c_{i+1} + c_i \\ c_i - d \end{bmatrix} \begin{bmatrix} a_{i+1} + c_{i+1} - c_i \\ c_{i+1} - d \end{bmatrix} v^{(c_i-d)b_i + (c_{i+1}-d)a_{i+1}} c^0(L, J_s).$$

This formula coincides with Zelevinski's inductive formula ([Z], Theorem 2) for the coefficients of Kazhdan-Lusztig polynomials for Grassmannians, where his variable t is our v . Note that Zelevinski's formula has a power of v different from ours, due to his use of shifted quantum binomials, while our quantum binomials are balanced.

Observe that formula (3.16) for Kazhdan-Lusztig polynomials is local. Formulas given by Theorems 3.4 and 2.5 for the Kazhdan-Lusztig basis have the advantage of being global and they make obvious the central role played by the quantum group $U_q \mathfrak{sl}_2$ and its intertwiners for the Kazhdan-Lusztig theory in the Grassmannian case.

ACKNOWLEDGEMENTS

The works of the authors was partially supported by National Science Foundations Grants #DMS-9700765 (first author), #DMS-9610201 (second author).

A.K. would like to thank G. Lusztig for fruitful discussions, and M.K. is grateful to V. Protsak for interesting discussions.

REFERENCES

- [CP] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge Univ. Press, Cambridge.
- [D] V. V. Deodhar, *On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials*, Journal of Algebra **111** (1987), 483-506.
- [FK] I. B. Frenkel and M. Khovanov, *Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$* , Duke Math. J. **87** (1997), 409-480.
- [GL] I. Grojnowski and G. Lusztig, *On basis of irreducible representations of quantum GL_n* , Contemp. Math., **139** (1992), 167-174.
- [J] M. A. Jimbo, *A q -analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), 247-252.
- [K] M. Khovanov, *Graphical calculus, canonical bases and Kazhdan-Lusztig theory*, Thesis, Yale University (1997).
- [KL1] D. Kazhdan, G. Lusztig, *Representation of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.
- [KL2] ———, *Schubert varieties and Poincare duality*, Proceedings of Symposia in Pure Mathematics, vol. 36 (1980), 185-203.
- [L] G. Lusztig, *Introduction to Quantum Groups*, Birkhauser, Boston, 1993.
- [LS] A. Lascoux, M.-P. Schützenberger, *Polynomes de Kazhdan & Lusztig pour les grassmanniennes*, Asterisque **87-88** (1981), 249-266.
- [Z] A. V. Zelevinski, *Small resolutions of singularities of Schubert varieties*, Funktsional. Anal. i Prilozhen. **17** (1983), 75-77. (Russian)

DEPT. OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06520-8283, USA

SCHOOL OF MATH., INST. ADV. STUDY, OLDEN LANE, PRINCETON, NJ, 08540, USA

E-mail address: mikhailek@math.ias.edu

MATH. DEPT., MIT, RM. 2-179, CAMBRIDGE, MA 02139, USA

E-mail address: kirillov@math.mit.edu